Theorem:- A series $\sum u_n$ converges iff to each $\varepsilon > 0$, there exists a positive integer m such that
$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon, \ \forall n \geq m \text{ and } p \geq 1.$$

Proof:- By Cauchy's principle of convergence, the sequence $\{S_n\}$ of partial sums of $\sum u_n$ converges iff to each $\varepsilon > 0$, $\exists$ a positive integer m, such that
$$|S_{n+p} - S_n| < \varepsilon, \ \forall n \geq m \text{ and } p \geq 1$$

$$\text{or, } |u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon, \ \forall n \geq m, p \geq 1.$$

Example:- Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{n^2+5}$ is convergent or divergent.

Ans:-
$$\because u_n = \frac{(-1)^n \cdot n^2}{n^2+5}$$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^2}} = 1 \neq 0$$

$\therefore$ The series is divergent.

Example: Show that the series $\sum \frac{1}{n}$ does not converge.

Proof:-
Suppose, if possible, the series converges.

Therefore, for any given ε, $\exists$ a positive integer m such that
$$\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right| < \varepsilon, \ \forall n \geq m \text{ and } p \geq 1$$

In particular, if $n = m$ and $p = m$, we get
$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$
$$> m \cdot \frac{1}{2m} = \frac{1}{2} > \varepsilon$$

This contradicts to our supposition, due to which the given series does not converge.

Note:- ~~It may be also seen $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$, the series~~ does not converge

Given $\varepsilon > 0$.

We must ...

Example: show that the Seri.

Second Method

$$\Sigma u_n = \Sigma \frac{1}{n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$\therefore$ The series does not converge.

Example:- Examine the Convergence of the Series

$$1^2 + 3^2 + 5^2 + 7^2 + \dots$$

$$\therefore u_n = (2n-1)^2$$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} (2n-1)^2 = \infty$$

$\therefore$ So, it is divergent.

Example:- Examine the Convergence of Series

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

$$\therefore S_n = \frac{1}{(t_n \text{ of } 1,3,5,\dots)\cdot(t_n \text{ of } 3,5,7,\dots)}$$

$$= \frac{1}{[1+(n-1)\times 2][3+(n-1)\times 2]}$$

$$= \frac{1}{(1+2n-2)(3+2n-2)} = \frac{1}{(2n-1)(2n+1)}$$

$$S_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$n=1, \quad S_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$$

$$n=2, \quad S_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$n=3, \quad S_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$S_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$S_1 + S_2 + S_3 + \dots + S_n = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2n+1}\right) = \frac{1}{2}(1-0) = \frac{1}{2}$$

$\therefore$ The series is convergent.

Theorem: Cauchy's General Principle of Convergence

Statement:- The necessary and sufficient condition for the convergence of a sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers, is that for given $\varepsilon > 0$, there exists a positive integer m such that $|S_{n+p} - S_n| < \varepsilon$, $n \geq m$, $p \geq 1$.

Proof:- (i) The necessary condition:—

Suppose, $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of real number that converges to l.

Hence, for given $\varepsilon > 0$, there exists a positive integer $m > n$ such that, $|S_n - l| < \frac{\varepsilon}{2}$ — (1), $n \geq m$

If $p \geq 1$ then $n+p > n \geq m$

Hence eq (1) becomes

$$|S_{n+p} - l| < \varepsilon/2 \quad \text{— (11)} \quad n \geq m, p \geq 1$$

Now, $|S_{n+1} - S_n| = |S_{n+p} - \cancel{l} + \cancel{l} - S_n|$

$$\leq |S_{n+p} - l| + |S_n - l|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (n \geq m, p \geq 1)$$

(ii) The sufficient condition:—

Suppose a positive integer $m > n$ such that for given $\varepsilon > 0$

$$|S_{n+p} - S_n| < \varepsilon \text{ — (3)}, \quad n \geq m, p \geq 1$$

We have to prove, $\{S_n\}_{n=1}^{\infty}$ is convergent.

Since, $\{S_n\}_{n=1}^{\infty}$ be a Cauchy's sequence of real numbers so, it is bounded.

From, Bolzano weierstrass theorem, each bounded sequence has at least one limit point.

so, $\{S_n\}_{n=1}^{\infty}$ converges to a limit point l.

From (3), for a given, $\varepsilon_1 = \frac{\varepsilon}{3} > 0$ such that

$$|S_{n+p} - S_n| < \frac{\varepsilon}{3}, \quad n \geq m, p \geq 1$$

Putting $n=m$, $|S_{m+p} - S_m| < \frac{\varepsilon}{3}$ —— (4), $p \geq 1$ #

Also, l is the limit point of $\{S_n\}_{n=1}^{\infty}$ ~~for~~ a positive integer $k > n$ must exist such that

$$|S_k - l| < \frac{\varepsilon}{3} \quad \text{—— (5)}$$

~~For~~ If $k > m$, eq (4) becomes.

$$|S_k - S_m| < \frac{\varepsilon}{3} \quad \text{—— (6)}$$

Now,

$$|S_{m+n} - l| = |S_{m+p} - S_m + S_m - S_k + S_k - l|$$

$$\leq |S_{m+p} - S_m| + |S_m - S_k| + |S_k - l|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\therefore |S_n - l| < \varepsilon, \quad \forall n \geq m$.

Thus, the sequence $\{S_n\}_{n=1}^{\infty}$ converges to l.